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Indecomposable representations of the Lorentz algebra in an angular momentum basis

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Abstract. Indecomposable (i.e. reducible but not completely reducible) representations of the (complex) algebra so(3, 1) of the Lorentz group are analysed. A master representation of so(3, 1) is obtained on the space of its universal enveloping algebra Ω. The basis chosen for so(3, 1) is the 'angular momentum' basis, and for Ω a 'natural basis' with respect to the angular momentum basis of so(3, 1). The master representation induces representations on a space Ω_- which is obtained from Ω as quotient space modulo certain ideals. These representations have the property that $\rho(h_3)$ is diagonal. A change of basis is made from the natural basis to the angular momentum basis and the representations on Ω_- are analysed in this new basis. The indecomposable (as well as the irreducible) representations which are obtained have the property that their so(3) content consists of *infinite-dimensional* indecomposable and irreducible so(3) representations. Certain of the indecomposable so(3, 1) representations have additional invariant subspaces which lead to quotient spaces which are of finite dimension. These quotient spaces then carry the familiar finite-dimensional representations of the Lorentz algebra. It is shown that the standard theory of representations of the Lorentz algebra is contained in this analysis as a special case of representations which are induced on certain quotient spaces of the space Ω_+.

1. Introduction

In a previous article (Gruber 1982) we investigated indecomposable representations of the complexification D_2 of the algebra so(3, 1) of the Lorentz group. This was done on the space of the universal enveloping algebra Ω of the semisimple Lie algebra D_2. The basis which was chosen for the space was a *natural basis*. That is, the basis elements of Ω were chosen as the set of ordered tensor products of the basis elements of the algebra D_2. Once the basis for D_2 has been specified, different natural bases will differ only in the ordering which is chosen for the basis elements of Ω.

The choice of a natural basis simplifies, in general, the analysis. However, the natural basis may not be the one which is of significance for different physical problems. While the natural basis has, for example, physical significance in problems which involve spin–quasispin (Judd 1968, Thomas and Gruber 1980) (seniority and particle number) as it does in the various shell models (atomic, nuclear), the natural basis may not be very suitable for other physical applications.

Apart from natural bases which have mathematical and some physical significance, the most important bases encountered in physical applications are probably angular momentum bases. These are bases of Ω which have the property that under restriction

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of so(3, 1) to its (angular momentum) subgroup so(3) the basis elements of Ω transform like states of angular momentum representations. The set of basis elements of Ω separates into subsets of bases of so(3) representations which are infinite dimensional, irreducible or indecomposable. The finite-dimensional irreducible representations of so(3) are, in turn, realised on quotient spaces of certain of the indecomposable representations with respect to so(3) invariant subspaces. The analysis given in this article thus differs from the analysis given by Gel'fand and Ponomarev (1968) whose analysis is carried out on spaces which have the property that they contain, under restriction to the so(3) subgroup, only finite-dimensional (irreducible) representations of so(3) (with finite multiplicity).

In this article we start out in § 3 with deriving a master representation for the complex Lorentz algebra so(3, 1) on the space of its universal enveloping algebra Ω (Humphreys 1972, Dixmier 1978). The basis chosen for so(3, 1) is an angular momentum basis, i.e. under restriction to so(3) one obtains the familiar angular momentum algebra. The basis chosen for Ω is a natural basis. In this article we first choose the particular ordering as given by $p_-h_-p_+h_+p_3h_3$. The master representation turns out to be indecomposable. Certain of its invariant subspaces are identified.

In § 4 we discuss a certain class of representations of the Lorentz algebra so(3, 1). This class of representations is defined by the conditions

$$\rho(h_3)\mathbb{1}=\Lambda_1\mathbb{1}, \qquad \rho(p_3)\mathbb{1}=\Lambda_2\mathbb{1}, \qquad \Lambda_1, \Lambda_2 \in \mathbb{C},$$
$$\rho(h_+)\mathbb{1}=\rho(p_+)\mathbb{1}=0.$$

We discuss in this section representations which are induced by the master representation on a space Ω_- which is equivalent to the space Ω modulo the left ideals which are defined by the conditions given above. While this is still done in a natural basis for Ω and Ω_- we switch now to an angular momentum basis for Ω_-. This is achieved through a determination of the $\rho(h_+)$ extremal vectors of Ω_-. One then obtains the relations (4.18). These relations can involve singular coefficients, depending on the value of the (complex) parameter Λ_1. However, even if the parameter Λ_1 is such that a 'singular band' of coefficients occurs, the relations nevertheless lead to so(3, 1) representations. In the case of a singular band these representations are subduced on invariant subspaces or induced on quotient spaces with the singular band lying outside of the representation space.

In cases A and B the relations (4.18) are discussed for specific values of the parameters Λ_1, Λ_2. The particular choice made for these parameters in cases A and B is motivated by our interest in analysing *indecomposable* representations of so(3, 1). All infinite-dimensional representations (indecomposable or irreducible) which are obtained on invariant subspaces or on quotient spaces of Ω_- have an so(3) content which consists of infinite-dimensional (indecomposable or irreducible) so(3) representations. Several types of indecomposable so(3, 1) representations are obtained for either case A or B. It turns out that if the so(3) content of an (infinite-dimensional) indecomposable so(3, 1) representation consists of (infinite-dimensional) *indecomposable* so(3) representation then the so(3, 1) representation itself has an additional invariant subspace. The quotient space with respect to this invariant subspace is finite dimensional and one obtains the familiar finite-dimensional representations of the Lorentz algebra (Gelfand *et al* 1963, Naimark 1964) as a special case of the analysis given in this article. Figures 3 and 4 serve as a kind of graphical summary of cases A and B.

In § 5 a basis for Ω is used where the ordering is chosen to be $p_+h_+p_-h_-p_3h_3$. This leads to representations on the space Ω_+. It is shown that the representations on Ω_- and Ω_+ are related through a Lie algebra automorphism. The representations ρ' on Ω_+ are of interest since they contain, in a direct manner, the standard theory of irreducible (finite- and infinite-dimensional) representations of the Lorentz algebra as a special case.

2. Notation and definitions

The angular momentum basis for the (complex) Lie algebra D_2 is given by the basis elements

$$D_2\colon \{h_3, h_+, h_-, p_3, p_+, p_-\} \tag{2.1}$$

with the following Lie products

$$\begin{aligned}
&[h_3, h_\pm] = \pm h_\pm, && [p_3, p_\pm] = \mp h_\pm,\\
&[h_3, p_\pm] = \pm p_\pm, && [p_3, h_\pm] = \pm p_\pm,\\
&[h_+, p_-] = [p_+, h_-] = 2p_3, && [p_+, h_+] = [p_-, h_-] = [p_3, h_3] = 0,\\
&[h_+, h_-] = [p_-, p_+] = 2h_3. &&
\end{aligned} \tag{2.2}$$

The usual restriction of the complex parameters then yields the real non-compact Lie algebra so(3, 1).

In the subsequent sections the following relations will be used:

$$\begin{aligned}
&[h_3, h_\pm^m] = \pm m h_\pm^m, \qquad [h_3, p_\pm^m] = \pm m p_\pm^m,\\
&[p_3, h_\pm^m] = \pm m p_\pm h_\pm^{m-1}, \qquad [p_3, p_\pm^m] = \mp m h_\pm p_\pm^{m-1},\\
&[h_\mp, h_\pm^m] = \mp 2m h_\pm^{m-1} h_3 - m(m-1) h_\pm^{m-1},\\
&[h_\mp, p_\pm^m] = \mp 2m p_\pm^{m-1} p_3 + m(m-1) h_\pm p_\pm^{m-2},\\
&[p_\mp, p_\pm^m] = \pm 2m p_\pm^{m-1} h_3 + m(m-1) p_\pm^{m-1},\\
&[p_\pm, h_\mp^m] = \pm 2m h_\mp^{m-1} p_3 - m(m-1) p_\mp h_\mp^{m-2}.
\end{aligned} \tag{2.3}$$

The relations (2.3) are valid only if all upper signs or all lower signs are taken simultaneously.

The symbols $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{N}^-$, $\mathbb{Z}$ stand for the non-negative integers, the positive integers, the negative integers and the set of all integers respectively. The symbols $\mathbb{R}$ and $\mathbb{C}$ represent the sets of real and complex numbers.

Indecomposable means that a representation is reducible, but not completely reducible.

3. The master representation in the natural basis

In this section the master representation ρ is given. This representation is defined on the space of the universal enveloping algebra Ω of so(3, 1). The basis chosen for Ω is called the *natural basis* and is defined by the set

$$\Omega\colon \{X(u, m, s, n, t, r) \equiv p_-^u h_-^m p_+^s h_+^n p_3^t h_3^r,\ u, m, s, n, t, r \in \mathbb{N}\} \tag{3.1}$$

where the product is the ordered tensor product and $X(0,0,0,0,0,0)=\mathbb{1}$ denotes the identity operator. All representations which will be discussed in this article are obtained from this master representation. This includes infinite-dimensional indecomposable representations of various types, infinite-dimensional irreducible representations as well as finite-dimensional irreducible representations of so(3, 1).

Acting with the elements of the algebra so(3, 1) upon the basis elements by means of the tensor product, and re-expressing the resultant element in terms of the basis elements by making use of relations (2.3), one obtains for the master representation ρ:

$$
\begin{aligned}
\rho(h_3)X &= X(u,m,s,n,t,r+1)+(n+s-m-u)X(u,m,s,n,t,r),\\
\rho(p_3)X &= X(u,m,s,n,t+1,r)+nX(u,m,s+1,n-1,t,r)\\
&\quad -sX(u,m,s-1,n+1,t,r)-mX(u+1,m-1,s,n,t,r)\\
&\quad +uX(u-1,m+1,s,n,t,r),\\
\rho(h_-)X &= X(u,m+1,s,n,t,r), \qquad \rho(p_-)X = X(u+1,m,s,n,t,r),\\
\rho(h_+)X &= X(u,m,s,n+1,t,r)+2mX(u,m-1,s,n,t,r+1)\\
&\quad +2uX(u-1,m,s,n,t+1,r)+2unX(u-1,m,s+1,n-1,t,r)\\
&\quad +u(u-1)X(u-2,m+1,s,n,t,r)-2usX(u-1,m,s-1,n+1,t,r)\\
&\quad +m(2s+2n-2u-m+1)X(u,m-1,s,n,t,r),\\
\rho(p_+)X &= X(u,m,s+1,n,t,r)+2mX(u,m-1,s,n,t+1,r)\\
&\quad -2uX(u-1,m,s,n,t,r+1)+2mnX(u,m-1,s+1,n-1,t,r)\\
&\quad -m(m-1)X(u+1,m-2,s,n,t,r)-2msX(u,m-1,s-1,n+1,t,r)\\
&\quad +u(2m-2n-2s+u-1)X(u-1,m,s,n,t,r).
\end{aligned} \tag{3.2}
$$

The master representation ρ, equation (3.2), is indecomposable in t as well as in r. Moreover, it is indecomposable in the sum $s+n=N$, $N\in\mathbb{N}$. This follows from the fact that the values for t and r do not decrease in (3.2), and neither does the sum $s+n$. For given values $N, T, R\in\mathbb{N}$ the subset of basis elements

$$V(N,T,R)\colon \{X(u,m,s,n,T+t,R+r),\ u,m,s,n,t,r\in\mathbb{N},\ s+n\geqslant N\} \tag{3.3}$$

transforms invariantly. We denote by $V(N,T,R)$ the subspace of Ω which is the linear span of the basis elements given by (3.3). Note that $V(0,0,0)=\Omega$. The subspaces $V(N,T,R)$ thus transform invariantly under the action of the master representation ρ and ρ thus *subduces* (indecomposable) representations on these subspaces. Moreover, ρ *induces* representations on the quotient spaces $\Omega/V(N,T,R)$. The induced representations are obtained from (3.2) by formally setting equal to zero all basis elements

$$X(u,m,s,N-s+n,T+t,R+r)\to 0, \qquad \text{for } u,m,s,n,t,r\in\mathbb{N},$$

while the subduced representations are obtained through restriction of the representations ρ, equation (3.2), to the basis elements (3.3).

4. Representations on Ω_- in the angular momentum basis

In this section representations are analysed which are induced by the master representation ρ, equation (3.2), for $s+n=N=0$, subject to the conditions

$$\rho(h_3)\mathbb{1}=\Lambda_1\mathbb{1}, \qquad \rho(p_3)\mathbb{1}=\Lambda_2\mathbb{1}, \qquad \Lambda_1, \Lambda_2\in\mathbb{C}. \tag{4.1}$$

The natural basis for the representation space Ω_- can be chosen as

$$\Omega_-: \{X(u,m)\equiv p_-^u h_-^m,\ u, m\in\mathbb{N}\}. \tag{4.2}$$

The representation which is induced by the master representation ρ is obtained as

$$\begin{aligned}
&\rho(h_3)X(u,m)=(\Lambda_1-m-u)X(u,m),\\
&\rho(p_3)X(u,m)=\Lambda_2X(u,m)-mX(u+1,m-1)+uX(u-1,m+1),\\
&\rho(h_-)X(u,m)=X(u,m+1), \qquad \rho(p_-)X(u,m)=X(u+1,m),\\
&\rho(h_+)X(u,m)=m(2\Lambda_1-2u-m+1)X(u,m-1)\\
&\qquad\qquad +2u\Lambda_2X(u-1,m)+u(u-1)X(u-2,m+1),\\
&\rho(p_+)X(u,m)=u(-2\Lambda_1+2m+u-1)X(u-1,m)\\
&\qquad\qquad +2m\Lambda_2X(u,m-1)-m(m-1)X(u+1,m-2).
\end{aligned} \tag{4.3}$$

These equations show that the basis $X(u,m)$ is not an angular momentum basis. It is therefore necessary to carry out a change of basis from the natural basis $X(u,m)$ to an angular momentum basis. The angular momentum basis is obtained by requiring that

$$\rho(h_+)y_N=0, \tag{4.4}$$

where

$$y_N=\sum_{k=0}^{N} c_kX(N-k,k) \tag{4.5}$$

and $\rho(h_3)y_N=(\Lambda_1-N)y_N$. The new basis

$$\Omega_-: \{y_N^m\equiv h_-^m y_N,\ m, N\in\mathbb{N}\} \tag{4.6}$$

is the desired angular momentum basis for Ω_-. That is, it holds in this basis that

$$\begin{aligned}
&\rho(h_3)y_N^m=(\Lambda_1-N-m)y_N^m, \qquad \rho(h_+)y_N^m=m[2(\Lambda_1-N)-m+1]y_N^{m-1},\\
&\rho(h_-)y_N^m=y_N^{m+1}.
\end{aligned} \tag{4.7}$$

These equations are equivalent to the familiar angular momentum representations. The following redefinition of basis elements

$$\begin{aligned}
&\Lambda_1-N=l, \qquad l-m=s, \qquad l=\tfrac{1}{2}k,\ k\in\mathbb{N}, \qquad |l,-l\rangle\equiv y_N^{2l},\\
&|l,s\rangle=\left(\prod_{k=-l}^{s-1}(l-k)(l+k+1)\right)^{1/2} y_N^{l-s}, \qquad s=l, l-1, \ldots, -l+1,
\end{aligned} \tag{4.8}$$

brings (4.7) into the familiar form

$$\begin{aligned}
&\rho(h_3)|l,s\rangle=s|l,s\rangle,\\
&\rho(h_+)|l,s\rangle=[(l-s)(l+s+1)]^{1/2}|l,s+1\rangle,\\
&\rho(h_-)|l,s\rangle=[(l-s+1)(l+s)]^{1/2}|l,s-1\rangle.
\end{aligned} \tag{4.9}$$

In (4.8) and (4.9) we have to place a restriction on the generally complex number Λ_1. Namely, Λ_1 is to be a real integer or half integer number, and moreover, $\Lambda_1 - N \geq 0$. The restriction of Λ_1 to these values yields the familiar irreducible finite-dimensional representations of so(3). In the following we will *not limit* our considerations to these special representations of so(3), but also consider indecomposable representations of so(3). Thus Λ_1 may be *any complex number* in what follows. This then requires that we use the form of the angular momentum representations given by (4.7) as in these equations Λ_1 can be an arbitrary complex number.

Before we discuss the $\rho(h_+)$ extremal vectors y_N it may be of advantage to illustrate the methods employed by means of the following discussion. This discussion will also lead to a special class of so(3, 1) representations (irreducible as well as indecomposable ones).

For $N = 0, 1, 2$ one obtains the $\rho(h_+)$ extremal vectors y_0, y_1 and y_2 as

$$y_0 = \mathbb{1}, \qquad y_1 = \boldsymbol{X}(1, 0) - (\Lambda_2/\Lambda_1)\boldsymbol{X}(0, 1),$$

$$y_2 = \boldsymbol{X}(2, 0) - [2\Lambda_2/(\Lambda_1 - 1)]\boldsymbol{X}(1, 1) - (2\Lambda_1 - 1)^{-1}[1 - 2\Lambda_2^2/(\Lambda_1 - 1)]\boldsymbol{X}(0, 2). \tag{4.10}$$

The spaces associated with the so(3) bases

$$\{y_0^m \equiv h_-^m y_0\}, \qquad \{y_1^m \equiv h_-^m y_1\}, \qquad \{y_2^m \equiv h_-^m y_2\}, \qquad m \in \mathbb{N}, \tag{4.11}$$

carry so(3) representations, given by (4.7). These representations are infinite dimensional and may be irreducible or indecomposable. If they are indecomposable, then (4.7) induces finite-dimensional irreducible so(3) representations on the quotient spaces.

Next one needs to calculate the action of the remaining operators of so(3,1). One obtains

$$\begin{aligned}
\rho(p_3)y_0^m &= (\Lambda_2/\Lambda_1)(\Lambda_1 - m)y_0^m - my_1^{m-1},\\
\rho(p_+)y_0^m &= m(\Lambda_2/\Lambda_1)(2\Lambda_1 - m + 1)y_0^{m-1} - m(m-1)y_1^{m-2},\\
\rho(p_-)y_0^m &= (\Lambda_2/\Lambda_1)y_0^{m+1} + y_1^m,\\
\rho(p_3)y_1^m &= \frac{(\Lambda_1^2 + \Lambda_2^2)(2\Lambda_1 - m - 1)}{\Lambda_1^2(2\Lambda_1 - 1)}y_0^{m+1} + \frac{\Lambda_2(\Lambda_1 + 1)(\Lambda_1 - m - 1)}{\Lambda_1(\Lambda_1 - 1)}y_1^m - my_2^{m-1},\\
\rho(p_+)y_1^m &= -\frac{(\Lambda_1^2 + \Lambda_2^2)(2\Lambda_1 - m - 1)(2\Lambda_1 - m)}{\Lambda_1^2(2\Lambda_1 - 1)}y_0^m\\
&\quad + \frac{m\Lambda_2(\Lambda_1 + 1)(2\Lambda_1 - m - 1)}{\Lambda_1(\Lambda_1 - 1)}y_1^{m-1} - m(m-1)y_2^{m-2},\\
\rho(p_-)y_1^m &= \frac{\Lambda_1^2 + \Lambda_2^2}{\Lambda_1^2(2\Lambda_1 - 1)}y_0^{m+2} + \frac{\Lambda_2(\Lambda_1 + 1)}{\Lambda_1(\Lambda_1 - 1)}y_1^{m+1} + y_2^m.
\end{aligned} \tag{4.12}$$

Inspecting (4.3) one sees that in general the action of $\rho(p)$ upon a basis element y_N^m yields a linear combination of the form $d_1 y_{N-1} + d_2 y_N + d_3 y_{N+1}$. From this observation it follows that if all the d_1 are zero (for y_N^m) with N fixed and m arbitrary, then the set of basis vectors

$$\{y_{N+s}^m, s, m \in \mathbb{N}\} \tag{4.13}$$

forms the basis for an *invariant* subspace. The representation then subduces a subrepresentation on this subspace. For the quotient space with respect to the invariant subspace a basis can be chosen as

$$\{y_i^m, i = 0, 1, \ldots, N-1; m \in \mathbb{N}\}. \tag{4.14}$$

The representation induces a representation on this quotient space. In (4.12) the d_1 become all zero for $\Lambda_1^2 + \Lambda_2^2 = 0$. We obtain then on the quotient space a representation (by formally setting equal to zero all basis elements in (4.12) except the y_0^m) which is given by the relations

$$\begin{aligned}
&\rho(h_3)y_0^m = (\Lambda_1 - m)y_0^m, \qquad \rho(h_+)y_0^m = m(2\Lambda_1 - m + 1)y_0^{m-1}, \qquad \rho(h_-)y_0^m = y_0^{m+1},\\
&\rho(p_3)y_0^m = \mathrm{i}(\Lambda_1 - m)y_0^m, \qquad \rho(p_+)y_0^m = \mathrm{i}m(2\Lambda_1 - m + 1)y_0^{m-1}, \qquad \rho(p_-)y_0^m = \mathrm{i}y_0^{m+1},
\end{aligned} \tag{4.15}$$

with $\Lambda_2 = \mathrm{i}\Lambda_1$ and basis $\{y_0^m, m \in \mathbb{N}\}$. This representation has the following properties:

(1) It is given in the angular momentum basis.

(2) $\mathrm{i}\rho(h) = \rho(p)$ holds. Thus it follows that *all* so(3) representations can be made so(3, 1) representations in this manner. We refer to Gruber and Klimyk (1984) for the so(3) representations.

(3) The representation is indecomposable for values Λ_1 such that

$$\Lambda_1 = \tfrac{1}{2}(m-1) \equiv l, \qquad m \in \mathbb{N}^+.$$

The subset $\{y_0^{2\Lambda_1+1+n}, n \in \mathbb{N}\}$ transforms irreducibly and the representation (4.15) subduces infinite-dimensional irreducible representations which are bounded above. $\rho(h_+)y_0^{2\Lambda_1+1} = 0$ and $\rho(p_+)y_0^{2\Lambda_1+1} = 0$ holds. The subduced representations are obtained from (4.15) through restriction to the invariant subset

$$\begin{aligned}
&m = 2l + 1 + n, \qquad \Lambda_1 = l, \qquad z^n \equiv y_0^{2l+1+n}, \qquad n \in \mathbb{N},\\
&\rho(h_3)z^n = -(l + 1 + n)z^n, \qquad \rho(h_+)z^n = -n(2l + 1 + n)z^{n-1}, \qquad \rho(h_-)z^n = z^{n+1},
\end{aligned} \tag{4.16}$$

and $\mathrm{i}\rho(h) = \rho(p)$.

For the quotient space with respect to the invariant subspace (the linear span of the subset) a basis can be chosen as

$$\{y_0^m, m = 0, 1, 2, \ldots, 2\Lambda_1 = 2l\}.$$

The representation which is induced on this quotient space is obtained as (setting equal to zero all basis elements y_N^m except y_0^m, $m = 0, 1, 2, \ldots, 2l$)

$$\begin{aligned}
&\rho(h_3)y_0^m = (l - m)y_0^m, \qquad \rho(h_+)y_0^m = m(2l - m + 1)y_0^{m-1},\\
&\rho(h_-)y_0^m = y_0^{m+1}, \qquad m = 0, 1, 2, \ldots, 2l,\\
&\rho(p) = \mathrm{i}\rho(h).
\end{aligned}$$

(Note, y_0^{2l+1} is an element of the invariant subspace, thus $y_0^{2l+1} \to 0$.)

These are the finite-dimensional irreducible representations of so(3, 1) of dimension $2l + 1$. The change of basis equation (4.8) brings these into the familiar form.

(4) For all values $\Lambda_1 \neq \tfrac{1}{2}m$, $m \in \mathbb{N}$, the representation is infinite dimensional, irreducible and bounded above. The only extremal vector is $\mathbb{1}$, i.e. $\rho(h_+)\mathbb{1} = \rho(p_+)\mathbb{1} = 0$.

(5) Replacing i by $-\mathrm{i}$ in (4.15) takes care of the case $\Lambda_2 = -\mathrm{i}\Lambda_1$.

Having discussed the special class of so(3, 1) representations which can be obtained from (irreducible as well as indecomposable) so(3) representations by 'substitution' we discuss now the general case of so(3, 1) representations in the angular momentum basis.

In order to obtain the formulae for the general case of so(3, 1) representations in the angular momentum basis it is necessary to study the properties of the $\rho(h_+)$ extremal vectors y_N, equation (4.5). It can be observed that the coefficients c_k of an extremal vector y_N satisfy the following recurrence relations,

$$
\begin{aligned}
c_1 &= (\Lambda_2 N/-\Lambda_1)c_0 \qquad \text{for } k=1,\\
c_k &= \frac{(N-k+1)(N-k+2)}{2k\Lambda_1-k(-2N+k+1)}c_{k-2}+\frac{2\Lambda_2(N-k+1)}{-2k\Lambda_1+k(-2N+k+1)}c_{k-1} \qquad \text{for } k\geqslant 2,
\end{aligned}
\tag{4.17}
$$

where c_0 is an undetermined constant. We set $c_0=1$. Making use of the relations (4.17) to determine the first few extremal vectors y_N it can be shown, by induction, that the master representation ρ, equation (3.2), induces the following relations on Ω_- with respect to the angular momentum basis:

$$
\begin{aligned}
&\Omega_-: \{y_N^m = h_-^m y_N,\ m, N\in\mathbb{N}\},\\
&\rho(h_3)y_N^m = (\Lambda_1-N-m)y_N^m,\\
&\rho(h_+)y_N^m = m[2(\Lambda_1-N)+1-m]y_N^{m-1}, \qquad \rho(h_-)y_N^m = y_N^{m+1},\\
&\rho(p_3)y_N^m = \alpha_N[2(\Lambda_1-N)+1-m]y_{N-1}^{m+1}+\beta_N(\Lambda_1-N-m)y_N^m - m y_{N+1}^{m-1},\\
&\rho(p_+)y_N^m = -\alpha_N[2(\Lambda_1-N)+1-m][2(\Lambda_1-N)+2-m]y_{N-1}^m\\
&\qquad\qquad +\beta_N m[2(\Lambda_1-N)+1-m]y_N^{m-1} - m(m-1)y_{N+1}^{m-2},\\
&\rho(p_-)y_N^m = \alpha_N y_{N-1}^{m+2}+\beta_N y_N^{m+1}+y_{N+1}^m,\\
&\alpha_N = [\Lambda_2^2+(\Lambda_1+1-N)^2]N[2(\Lambda_1+1)-N]/(\Lambda_1+1-N)^2[2(\Lambda_1-N)+3][2(\Lambda_1-N)+1],\\
&\beta_N = \Lambda_2(\Lambda_1+1)/(\Lambda_1-N)(\Lambda_1+1-N),\\
&\Lambda_1, \Lambda_2\in\mathbb{C}, \qquad N, m\in\mathbb{N}.
\end{aligned}
\tag{4.18}
$$

In equations (4.18), Λ_1 and Λ_2 represent *arbitrary complex numbers*. It must be noted, however, that for certain values of Λ_1 the coefficients α_N and β_N may become singular. Equations (4.18) may nevertheless define representations, either on invariant subspaces or on quotient spaces, or on both. What happens depends on the order in which the limits are taken in the coefficients α_N and β_N. In the following we discuss (4.18) for special values of the parameters Λ_1 and Λ_2.

Case A:

$$\Lambda_1=\tfrac{1}{2}M, \qquad \Lambda_2=\pm\tfrac{1}{2}\mathrm{i}n, \qquad M, n \text{ odd integers.}$$

The denominator of α_N becomes zero for $N=\frac{1}{2}(M+1), \frac{1}{2}(M+3)$. The numerator of α_N becomes zero for $N=0, N=M+2$ and $N=\pm\frac{1}{2}n+\frac{1}{2}M+1$. The numerator of β_N becomes zero for $n=0$. The coefficients α_N can become undetermined and the existence of finite limits needs to be investigated. The vanishing of α_N signifies an invariant subspace. In the following we consider four subcases.

(A1) $\qquad \Lambda_1=\frac{1}{2}, \qquad \Lambda_2=\pm\frac{1}{2}\mathrm{i} \qquad$ (i.e. $M=n=1$).

For this case $\alpha_0=\alpha_3=0$. The value of the coefficients α_1 and α_2 depends on the order in which N and the limits of Λ_1 and Λ_2 are taken. If we characterise the order by writing the symbols in parentheses, with the understanding that the sequence is from right to left, then we obtain

		α_1	α_2
I	$(\Lambda_2, \Lambda_1, N)$, $(\Lambda_2, N, \Lambda_1)$	$\pm\infty$	$\pm\infty$
II	$(\Lambda_1, \Lambda_2, N)$, $(\Lambda_1, N, \Lambda_2)$, $(N, \Lambda_1, \Lambda_2)$, $(N, \Lambda_2, \Lambda_1)$	2	2
III	$(N, \Lambda_1, \Lambda_2 \to \pm i\Lambda_1)$, $(N, \Lambda_2, \Lambda_1 \to \mp i\Lambda_2)$, $(\Lambda_1, \Lambda_2 \to \pm i\Lambda_1, N)$, $(\Lambda_1, N, \Lambda_2 \to \pm i\Lambda_1)$, $(\Lambda_2, N, \Lambda_1 \to \mp i\Lambda_2)$, $(\Lambda_2, \Lambda_1 \to \mp i\Lambda_2, N)$	0	4

(4.19)

For case I the relations (4.18) define an irreducible representation of so(3, 1) on the subspace with basis

$$V_3: \{y^m_{3+N}, m, N \in \mathbb{N}\}.$$

The so(3) content of this representation consists of the infinite-dimensional irreducible representations with highest weights $-\frac{5}{2}, -\frac{7}{2}, -\frac{9}{2}, \ldots$.

For cases II and III equations (4.18) subduce on the invariant subspace V_3 the same so(3, 1) representation as for case I. In cases II and III, however, the representation (4.18) also induces a representation on the quotient space with basis

$$Q_3 = \Omega_-/V_3: \{y^m_0, y^m_1, y^m_2, m \in \mathbb{N}\}.$$

The representation which is induced by (4.18) on Ω_-/V_3 is indecomposable for both cases II and III, though in a different manner for each case.

(A2) $\Lambda_1 = \frac{1}{2}M, \qquad \Lambda_2 = \pm\frac{1}{2}iM, \qquad M = 3, 5, 7, \ldots$.

For this case $\alpha_0 = \alpha_1 = \alpha_{M+1} = \alpha_{M+2} = 0$, while α_N becomes singular for $N = \frac{1}{2}(M+1), \frac{1}{2}(M+3)$. The equation (4.18) defines on the subspace W_{M+1} with basis

$$W_{M+1}: \{y^m_{M+1+N}, N, m \in \mathbb{N}\} \tag{4.20}$$

an indecomposable so(3, 1) representation. The basis for its invariant subspace can be chosen as

$$W_{M+2}: \{y^m_{M+2+N}, N, m \in \mathbb{N}\}. \tag{4.21}$$

The representation subduces an irreducible so(3, 1) representation on W_{M+2} whose so(3) content consists of the so(3) irreducible representations with highest weights $-\frac{1}{2}M - 2, -\frac{1}{2}M - 3, \ldots$. On the quotient space W_{M+1}/W_{M+2} equations (4.18) induce an irreducible so(3,1) representation with highest weight $-\frac{1}{2}M - 1$. The subspace spanned by

$$W_1: \{y^m_{1+N}, N, m \in \mathbb{N}\}$$

transforms invariantly under (4.18); however, it contains the singular band. On the quotient space Ω_-/W_1, however, (4.18) induce again an so(3, 1) representation. A basis for this space can be chosen as

$$\Omega_-/W_1\colon \{y_0^m,\ m \in \mathbb{N}\}. \tag{4.22}$$

This space carries an indecomposable so(3, 1) representation which does not decompose under restriction to so(3). The invariant subspace of this representation is spanned by

$$\{y_0^{M+1+m},\ m \in \mathbb{N}\} \tag{4.23}$$

and carries an (infinite) dimensional irreducible so(3, 1) representation with highest weight $-\frac{1}{2}M-1$. On the quotient space a finite-dimensional irreducible so(3, 1) representation is induced, with dimension $M+1$. The so(3) content is identical to the so(3, 1) representation (i.e. it does not decompose). We refer to the discussion following (4.15).

(A3) $\quad \Lambda_1 = \frac{1}{2}M, \qquad \Lambda_2 = \pm\frac{1}{2}\mathrm{i}, \qquad n = 1, M = 3, 5, 7, \ldots.$

In this case again care has to be taken regarding the order in which the value N and the limits for Λ_1 and Λ_2 are chosen. $\alpha_0 = \alpha_{M+2} = 0$ holds while for $\alpha_N, N = \frac{1}{2}(M+1), N = \frac{1}{2}(M+3)$ one obtains

		$N=\frac{1}{2}(M+1)$	$N=\frac{1}{2}(M+3)$
I	$(\Lambda_2, \Lambda_1, N)$ $(\Lambda_2, N, \Lambda_1)$	$\pm\infty$	$\pm\infty$
II	$(\Lambda_1, \Lambda_2, N), (\Lambda_1, N, \Lambda_2)$ $(N, \Lambda_1, \Lambda_2), (N, \Lambda_2, \Lambda_1)$	$\frac{1}{4}(M+1)(M+3)$	$\frac{1}{4}(M+1)(M+3)$
III	$(\Lambda_1, N, \Lambda_2 \to \pm\mathrm{i}(\Lambda_1 - N + 1)$ $(N, \Lambda_1, \Lambda_2 \to \pm\mathrm{i}(\Lambda_1 - N + 1)$ $(\Lambda_1, \Lambda_2 \to \pm\mathrm{i}(\Lambda_1 - N + 1), N)$ $(N, \Lambda_2 \to \pm\mathrm{i}(\Lambda_1 - N + 1), \Lambda_1)$	0	0

The vanishing of the coefficients α_N for $N = 0, M+2$ defines subspaces which remain invariant under the (possibly) singular transformations given by (4.18). The property $\alpha_{M+2} = 0$ defines an invariant subspace for both cases I and II for which a basis can be chosen as

$$V_{M+2}\colon \{y_{M+2+N}^m,\ N, m \in \mathbb{N}\}. \tag{4.24}$$

In case I equations (4.18) subduce an irreducible so(3, 1) representation on V_{M+2} whose so(3) content consists of the so(3) irreducible representations with highest weights $-\frac{1}{2}M-2, -\frac{1}{2}M-3, \ldots.$

In case II, equations (4.18) define on Ω_- an indecomposable so(3, 1) representation which contains V_{M+2} as invariant subspace as well as an invariant subspace with basis

$$V_M\colon \begin{cases} y_N^{M-2N+t}, & N = 0, 1, 2, \ldots, \frac{1}{2}(M-1);\ t \in \mathbb{N}^+, \\ y_N^m, & N = \frac{1}{2}(M+1), \frac{1}{2}(M+3), \ldots,\ m \in \mathbb{N}, \end{cases} \tag{4.25}$$

which in turn contains the subspace V_{M+2} as an invariant subspace. The so(3) content of this representation consists of the indecomposable so(3) representations with highest weights $\frac{1}{2}M, \frac{1}{2}M-1, \ldots, \frac{1}{2}$ and the so(3) irreducible representations with highest weights $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$. Equations (4.18) induce on the quotient space

$$\Omega_-/V_M\colon \{y_N^k, N=0,1,2,\ldots,\tfrac{1}{2}(M-1), k=0,1,2,\ldots,M-2N\} \tag{4.26}$$

a finite-dimensional irreducible representation of so(3, 1) whose dimension is given as

$$\dim \rho(M)=\tfrac{1}{4}(M+1)(M+3) \tag{4.27}$$

and its so(3) content consists of the finite-dimensional irreducible so(3) representations with highest weights $\frac{1}{2}M, \frac{1}{2}M-1, \ldots, \frac{1}{2}$.

It needs to be proven that the basis elements given by (4.25) span an so(3, 1) invariant subspace. This can be seen as follows. Consider the following array of six basis vectors:

$$\begin{array}{|ccc}
y_k^{M-2k+1} & y_k^{M-2k+2} & y_k^{M-2k+3} \\
\hline
y_{k-1}^{M-2k+1} & y_{k-1}^{M-2k+2} & \multicolumn{1}{|c}{y_{k-1}^{M-2k+3}}
\end{array} \tag{4.28}$$

where $k=0,1,2,\ldots,\frac{1}{2}(M-1), n=M-2k, M, n=1,3,5,\ldots$. It needs to be shown that the action of the representation upon any basis element which is to the right of (or above) the line cannot lead to elements which are to the left of (or below) the line. If this is done then we have found an invariant subspace (see also figures (1)–(3)). The action of the so(3) subalgebra is along each line separately (i.e. horizontal) and it is easily verified that y_k^{M-2k+1} and y_{k-1}^{M-2k+3} are $\rho(h_+)$ extremal vectors (i.e. they are mapped to zero and not to the left of the line). Moreover, it holds that

$$\begin{aligned}
&\rho(p_+)y_k^{M-2k+1}\sim y_{k+1}^{M-2k-1}, \qquad \rho(p_+)y_k^{M-2k+2}=c_1y_k^{M-2k+1}+c_2y_{k+1}^{M-2k},\\
&\rho(p_3)y_k^{M-2k+1}=d_1y_k^{M-2k+1}+d_2y_{k+1}^{M-2k},
\end{aligned} \tag{4.29}$$

since for these values the factors involving the m vanish in (4.18). The c's and d's denote non-vanishing coefficients. Thus, the action of $\rho(p_+)$ and $\rho(p_3)$ upon these elements does not map them upon elements to the left of the line. Inspection of (4.18) shows that the action of the remaining operators upon elements to the right of the line cannot lead to elements which are to the left of the line. Thus the elements to the right of the line correspond to an invariant subspace.

At this point it may be useful to consider an example. We choose $\Lambda_1=\frac{3}{2}, \Lambda_2=\pm\frac{1}{2}\mathrm{i}$. Equations (4.18) induce on the quotient space with basis $\{y_0^m, y_1^m\}$ the indecomposable

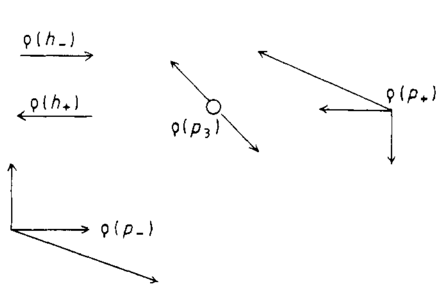

Figure 1. Action of the shift operators of so(3, 1) upon the elements of Ω_- in the angular momentum basis as plotted in figures 2–4. Note that this action is to be multiplied by matrix elements which may become zero.

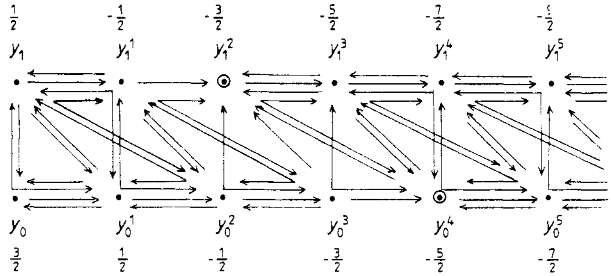

Figure 2. Graphical representation of the indecomposable so(3, 1) representation $\Lambda_1 = \frac{3}{2}$, $\Lambda_2 = \pm i/2$ ($m = 3$, $n = 1$). The arrows indicate the action of the various operators ρ. The action of $\rho(h_3)$ on a basis vector yields a multiple $\Lambda_1 - N - m$ of the basis vector, the action of $\rho(p_3)$ on a basis vector yields, with one exception (y_1^2), a linear combination of two basis vectors, one of which is the original state. In the figure we indicate the shift vectors only. The circled points represent $\rho(h_+)$ external vectors. On the invariant subspace $\{y_0^{4+m}, y_1^{2+m}, m \in \mathbb{N}\}$ one obtains an so(3, 1) irreducible representation with so(3) content of two infinite-dimensional irreducible representations with highest weights $-\frac{3}{2}$ and $-\frac{5}{2}$. On the quotient space $\{y_0, y_0^1, y_0^2, y_0^3, y_1, y_1^1\}$ with respect to the invariant subspace one obtains a six-dimensional irreducible so(3, 1) representation with its so(3) content consisting of the two finite-dimensional irreducible representations with highest weights $\frac{3}{2}$, $\frac{1}{2}$. The so(3, 1) indecomposable representation $\Lambda_1 = \frac{3}{2}$, $\Lambda_2 = i/2$ has an so(3) content which consists of the two so(3) indecomposable representations with highest weights $\frac{3}{2}$ and $\frac{1}{2}$.

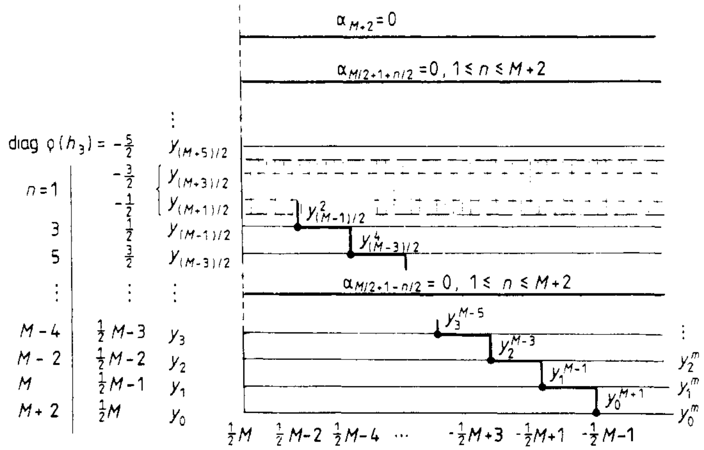

Figure 3. Graphical description for case A ($\Lambda_1 = \frac{1}{2}M$, $\Lambda_2 = \frac{1}{2}in$, M, $n = 1, 3, 5, \ldots$). The basis vectors are plotted in a rectangular array, with N increasing from bottom to top and m increasing from left to right. The dots represent $\rho(h_+)$ extremal vectors. The eigenvalues of $\rho(h_3)$ are given by $\frac{1}{2}M - N - m$. In the shaded area the coefficients α_N are singular (or undetermined). The coefficient $\alpha_N = 0$ (or is undetermined) for $N = \pm\frac{1}{2}n + \frac{1}{2}M + 1 \geq 0$. That is, for these values of N the elements y above (and including) the horizontal line which represents the y_N^m form an invariant subspace. For given $N = -\frac{1}{2}n + \frac{1}{2}M + 1$, $n \leq M$, a quotient space can be formed, a basis of which can be chosen as the set of elements represented by the lowest $N = -\frac{1}{2}n + \frac{1}{2}M + 1$ lines. The representation which is induced on this quotient space is indecomposable. All the elements to the right of the full line including the line span an invariant subspace. Forming the quotient space of the quotient space finite-dimensional irreducible representations are obtained. The elements to the left of the full line span the finite-dimensional space which carries the representation. The elements on the full line and to the right of it span the space for an infinite-dimensional irreducible representation. In general both signs $\pm\frac{1}{2}n$ need to be included in the discussion, as well as all other values for n. Whenever the singular region is not contained in an invariant subspace, or else in the quotient space, representations of so(3, 1) are obtained.

so(3, 1) representation

$$
\begin{aligned}
&\rho(h_3)y_0^m = (\tfrac{3}{2}-m)y_0^m, \qquad \rho(h_+)y_0^m = m(4-m)y_0^{m-1},\\
&\rho(h_-)y_0^m = y_0^{m+1}, \qquad \rho(p_3)y_0^m = \tfrac{1}{3}\mathrm{i}(\tfrac{3}{2}-m)y_0^m - my_1^{m-1},\\
&\rho(p_+)y_0^m = \tfrac{1}{3}\mathrm{i}m(4-m)y_0^{m-1} - m(m-1)y_1^{m-2},\\
&\rho(p_-)y_0^m = \tfrac{1}{3}\mathrm{i}y_0^{m+1} + y_1^m, \qquad \rho(h_3)y_1^m = (\tfrac{1}{2}-m)y_1^m,\\
&\rho(h_+)y_1^m = m(2-m)y_1^{m-1}, \qquad \rho(h_-)y_1^m = y_1^{m+1},\\
&\rho(p_3)y_1^m = \tfrac{4}{9}(2-m)y_0^{m+1} + \tfrac{5}{3}\mathrm{i}(\tfrac{1}{2}-m)y_1^m,\\
&\rho(p_+)y_1^m = -\tfrac{4}{9}(2-m)(3-m)y_0^m + \tfrac{5}{3}\mathrm{i}m(2-m)y_1^{m-1},\\
&\rho(p_-)y_1^m = \tfrac{4}{9}y_0^{m+2} + \tfrac{5}{3}\mathrm{i}y_1^{m+1}.
\end{aligned}
\tag{4.30}
$$

Figure 2 represents this representation in graphical form. The basis vectors to the right of the circled points (including those) span an infinite-dimensional invariant subspace. Restriction of the representation (4.30) to this subspace yields an infinite-dimensional irreducible so(3, 1) representation. Its so(3) content consists of two infinite-dimensional irreducible so(3) representations which are bounded above and have highest weights $-\frac{5}{2}$ and $-\frac{3}{2}$ respectively. The representation (4.30) induces on the quotient space with basis $\{y_0, y_0^1, y_0^2, y_0^3, y_1^0, y_1^1\}$ a six-dimensional irreducible so(3, 1) representation. Its so(3) content consists of the two spin representations with highest weights $\frac{3}{2}$ and $\frac{1}{2}$. This representation is obtained from (4.30) by formally setting

$$
y_0^m \to 0, \qquad m = 4, 5, 6, \ldots, \qquad y_1^m \to 0, \qquad m = 2, 3, 4, \ldots.
$$

In matrix form one then obtains

$$
\rho(h_3) = \begin{pmatrix}
\frac{3}{2} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{2} & 0 & 0 & 0 & 0\\
0 & 0 & -\frac{1}{2} & 0 & 0 & 0\\
0 & 0 & 0 & \frac{3}{2} & 0 & 0\\
0 & 0 & 0 & 0 & \frac{1}{2} & 0\\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}
\end{pmatrix}, \qquad
\rho(h_+) = \begin{pmatrix}
0 & 3 & 0 & 0 & 0 & 0\\
0 & 0 & 4 & 0 & 0 & 0\\
0 & 0 & 0 & 3 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\rho(h_-) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}, \qquad
\rho(p_3) = \begin{pmatrix}
\frac{1}{2}\mathrm{i} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{6}\mathrm{i} & 0 & 0 & \frac{8}{9} & 0\\
0 & 0 & -\frac{1}{6}\mathrm{i} & 0 & 0 & \frac{4}{9}\\
0 & 0 & 0 & -\frac{1}{2}\mathrm{i} & 0 & 0\\
0 & -1 & 0 & 0 & \frac{5}{6}\mathrm{i} & 0\\
0 & 0 & -2 & 0 & 0 & -\frac{5}{6}\mathrm{i}
\end{pmatrix},
$$

$$
\rho(p_+) = \begin{pmatrix}
0 & \mathrm{i} & 0 & 0 & -\frac{24}{9} & 0\\
0 & 0 & \frac{4}{3}\mathrm{i} & 0 & 0 & -\frac{8}{9}\\
0 & 0 & 0 & \mathrm{i} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -2 & 0 & 0 & \frac{5}{3}\mathrm{i}\\
0 & 0 & 0 & -6 & 0 & 0
\end{pmatrix}, \qquad
\rho(p_-) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{3}\mathrm{i} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1}{3}\mathrm{i} & 0 & 0 & \frac{4}{9} & 0\\
0 & 0 & \frac{1}{3}\mathrm{i} & 0 & 0 & \frac{4}{9}\\
1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & \frac{5}{3}\mathrm{i} & 0
\end{pmatrix}.
$$

(A4) $\Lambda_1 = \frac{1}{2}M$, $\Lambda_2 = \frac{1}{2}in$, $M, n = 3, 5, 7, \ldots ; n < M$.

In this case $\alpha_N = 0$ for $N = 0, M+2, \pm\frac{1}{2}n + \frac{1}{2}M + 1$ while α_N becomes singular for $N = \frac{1}{2}(M+1), \frac{1}{2}(M+3)$. The singular band lies in between $N = \pm\frac{1}{2}n + \frac{1}{2}M + 1$. See figure 3. $N = \frac{1}{2}n + \frac{1}{2}M + 1$ defines an invariant subspace which contains in turn an invariant subspace defined by $N = M + 2$. The relations (4.18) subduce and induce representations on these invariant subspaces and on the quotient space. The value $N = -\frac{1}{2}n + \frac{1}{2}M + 1$ defines another invariant subspace V_{Mn} with respect to the transformations (4.18) which contains the previous two subspaces as invariant subspaces, as well as the singular band. On the quotient space of Ω_- modulo the invariant subspace V_{Mn} with basis (the bottom $-\frac{1}{2}n + \frac{1}{2}M + 1$ lines of figure 3)

$$\Omega_-/V_{Mn}: \{y_s^m, s = 0, 1, 2, \ldots, \tfrac{1}{2}M - \tfrac{1}{2}n, m \in \mathbb{N}\}, \qquad (4.31)$$

the relations (4.18) induce an indecomposable so(3, 1) representation. Its invariant subspace is spanned by the set of basis elements

$$\{y_s^{M-2s+t}, s = 0, 1, 2, \ldots, \tfrac{1}{2}M - \tfrac{1}{2}n, t \in \mathbb{N}^+\} \qquad (4.32)$$

(the elements to the right of, and above, the full line, including the line). The representation which is subduced by (4.18) on this space is infinite dimensional and irreducible. Its so(3) content consists of the infinite-dimensional irreducible representations with highest weights $-\frac{1}{2}M - 1, -\frac{1}{2}M - 2, \ldots, \frac{1}{2}n - 1$. For the quotient space of the representation space (4.31) with respect to its invariant subspace a basis can be chosen as

$$\{y_s^m, s = 0, 1, \ldots, \tfrac{1}{2}M - \tfrac{1}{2}n, m = 0, 1, 2, \ldots, M - 2s\} \qquad (4.33)$$

(the elements represented by the first N lines to the left of, and below, the full line). The representation (4.18) induces on this space a finite-dimensional irreducible representation. Its dimension is given by

$$\dim \rho(M,n) = \tfrac{1}{4}(M+n)(M-n+2). \qquad (4.34)$$

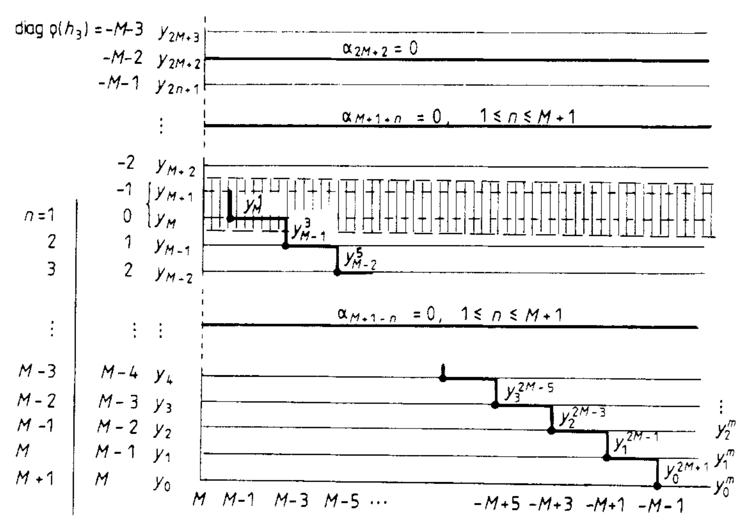

Figure 4. Graphical description for case B ($\Lambda_1 = M$, $\Lambda_2 = in$; $M, n \in \mathbb{N}$). The interpretation of this figure is analogous to figure 3. Some invariant subspaces are indicated by full lines. The areas to the right of and above the lines represent the invariant subspaces.

These finite-dimensional representations are easily obtained from (4.18) by formally setting to zero all elements of the invariant subspace (4.32).

Case B:

$$\Lambda_1 = M, \qquad \Lambda_2 = \pm \mathrm{i}n, \qquad M, n \in \mathbb{N}.$$

The denominator of α_N becomes zero for $N = M+1$. The denominator of β_N becomes zero for $N = M, M+1$. The numerator of α_N becomes zero for $N = 0$, $N = 2M+2$ and $N = \pm n + M + 1$. Again care has to be taken with the limits as these depend upon the order in which they are taken. We distinguish three subcases. See figures 1 and 4.

(B1) $\Lambda_1 = \Lambda_2 = 0$, i.e. $M = n = 0$.

For this case $\alpha_0 = \alpha_2 = 0$. The values of α_1, β_0 and β_1 depend upon the order of the limits. We obtain

		α_1	β_0	β_1
I	$(\Lambda_2, \Lambda_1, N)$ $(\Lambda_2, N, \Lambda_1)$	$\pm\infty$	$\pm\infty$	$\pm\infty$
II	$(\Lambda_1, \Lambda_2, N)$ $(\Lambda_1, N, \Lambda_2)$ $(N, \Lambda_1, \Lambda_2)$ $(N, \Lambda_2, \Lambda_1)$ $(N, \Lambda_2, \Lambda_1 \to \mp \mathrm{i}\Lambda_2)$ $(N, \Lambda_1, \Lambda_2 \to \pm \mathrm{i}\Lambda_1)$	-1	0	0
III	$(\Lambda_2, \Lambda_1 \to \mp \mathrm{i}\Lambda_2, N)$ $(\Lambda_2, N, \Lambda_1 \to \mp \mathrm{i}\Lambda_2)$ $(\Lambda_1, \Lambda_2 \to \pm \mathrm{i}\Lambda_1, N)$ $(\Lambda_1, N, \Lambda_2 \to \pm \mathrm{i}\Lambda_1)$	0	$\pm\mathrm{i}$	$\mp\mathrm{i}$

For case I equations (4.18) define an irreducible representation of so(3, 1) on the invariant subspace which is defined by $\alpha_2 = 0$. A basis for this subspace can be chosen as

$$V_2: \{y^m_{2+N}, m, N \in \mathbb{N}\}. \tag{4.35}$$

The so(3) content of the so(3, 1) representation which is obtained on V_2 is given by the irreducible so(3) representations with highest weights $-2, -3, -4, \ldots$.

For both cases II and III V_2 also forms an invariant subspace, and thus (4.18) subduce on V_2 the same (sub)representation as in case I. For case II one obtains in addition an irreducible so(3, 1) representation which is induced by (4.18) on the quotient space with basis

$$\Omega_-/V_2: \{y^m_0, y^m_1, m \in \mathbb{N}\}. \tag{4.36}$$

The so(3) content consists of the so(3) indecomposable representation with highest weight 0 and so(3) irreducible representations with highest weight -1.

For case III there exists an additional invariant subspace with basis

$$V_1: \{y^m_{1+N}, m, N \in \mathbb{N}\}. \tag{4.37}$$

The subspace V_1 contains in turn the subspace V_2, equation (4.35), as so(3, 1) invariant subspace. The representation (4.18) induces on the quotient space Ω_-/V_2, equation (4.36), an indecomposable representation, which has an invariant subspace whose basis can be chosen as

$$\{y_1^m, m \in \mathbb{N}\}. \tag{4.38}$$

On the quotient space and on the invariant subspace then irreducible so(3, 1) representations are induced and subduced. These so(3, 1) representations do not decompose under restriction to so(3). They have highest weights 0 and -1 respectively.

(B2) $\Lambda_1 = M, \quad \Lambda_2 = \pm i\Lambda_1, \quad M \in \mathbb{N}^+.$

For this case $\alpha_N = 0$ for $N = 0, 1, 2M+1$ and $2M+2$. The value for $\beta_0 = \pm i$. The coefficients α_N, $N = M+1$ and β_N, $N = M, M+1$ become singular. The singular band lies in between $N = 1$ and $N = 2M+1$.

The invariant subspace

$$V_{2M+2}: \{y_{2M+2+N}^m, m, N \in \mathbb{N}\} \tag{4.39}$$

carries an infinite-dimensional irreducible so(3, 1) representation. Its so(3) content consists of the so(3) irreducible infinite-dimensional representations with highest weights $-M-2, -M-3, -M-4, \ldots$. The invariant subspace

$$\{y_{2M+1+N}^m, m, N \in \mathbb{N}\} \tag{4.40}$$

carries an so(3, 1) indecomposable representation. On the quotient space with basis

$$\{y_{2M}^m, m \in \mathbb{N}\} \tag{4.41}$$

one obtains an so(3, 1) (and so(3)) irreducible representation with highest weight $-M$. The set of elements

$$\{y_N^m, N \in \mathbb{N}^+, m \in \mathbb{N}\} \tag{4.42}$$

can be chosen as a basis for the so(3, 1) invariant subspace defined by $\alpha_1 = 0$. Equations (4.18) yield singular elements in this space. For the quotient space Ω_- by the invariant subspace (4.42) a basis can be chosen as

$$\{y_0^m, m \in \mathbb{N}\}. \tag{4.43}$$

The representations which are induced on it are given by (4.15).

(B3) $\Lambda_1 = M, \quad \Lambda_2 = \pm in, \quad M \in \mathbb{N}^+, \quad 1 \leqslant n < M.$

For this case we have the same 'singular band' as in (B2). See figure 4. The coefficients α_N become zero for $N = 0, 2M+2, n+M+1, -n+M+1$. The invariant subspaces which correspond to the vanishings of these α's have basis

$$\{y_{2M+2+N}^m, m, N \in \mathbb{N}\}, \tag{4.44}$$

$$\{y_{M+1+n+N}^m, m, N \in \mathbb{N}\} \tag{4.45}$$

$$\{y_{M+1-n+N}^m, m, N \in \mathbb{N}\}. \tag{4.46}$$

The last one contains the 'singular band,' while each space contains the preceding subspaces as invariant subspaces. The entire space, Ω_-, and the subspace (4.46), contain an additional invariant subspace, which in turn contain both subspaces (4.45) and (4.44) as invariant subspaces. Just as in case (A3) the existence of these additional subspaces can be proven by considering the property of the array of the six vectors

$$\begin{array}{|lll} y_{N+1}^{2M-2N-1} & y_{N+1}^{2M-2N} & y_{N+1}^{2M-2N+1} \\ \hline y_{N}^{2M-2N-1} & y_{N}^{2M-2N} & \multicolumn{1}{|l}{y_{N}^{2M-2N+1}.} \end{array} \tag{4.47}$$

It holds that

$$\rho(p_3)y_{N+1}^{2M-2N-1} = c_1 y_{N+1}^{2M-2N-1} + c_2 y_{N+2}^{2M-2N-2}, \qquad \rho(p_+)y_{N+1}^{2M-2N-1} \sim y_{N+2}^{2M-2N-3},$$
$$\rho(p_+)y_{N+1}^{2M-2N} = d_1 y_{N+1}^{2M-2N-1} + d_2 y_{N+2}^{2M-2N-2}. \tag{4.48}$$

From (4.48) and from an inspection of the action of the other elements of the algebra upon the basis vectors, (4.18), it follows that the elements to the right of and above the full line form an invariant subspace (see figure 4). The basis elements of these additional invariant subspaces consist of the elements to the right of the full line and up. Since they contain the 'singular band' we will be interested only in quotient spaces which do not contain the singular band. The basis for the quotient spaces of interest is given by

$$\{y_N^m, N = 0, 1, 2, \ldots, M-n, m \in \mathbb{N}\} \tag{4.49}$$

with invariant subspace

$$\{y_N^{2M+1-2N+t}, N = 0, 1, \ldots, M-n, t \in \mathbb{N}\} \tag{4.50}$$

and quotient space

$$\{y_N^m, N = 0, 1, \ldots, M-n;\ m = 0, 1, \ldots, 2M-2N\}. \tag{4.51}$$

Equations (4.18) induce on the space equation (4.50) an infinite-dimensional irreducible so(3, 1) representation with its so(3) content consisting of the infinite-dimensional irreducible so(3) representations with highest weights $-M-1, -M-2, \ldots, -(n+1)$. On the quotient space (4.51) finite-dimensional irreducible so(3, 1) representations are obtained. Their dimension is

$$\dim \rho(M, n) = (M+n+1)(M-n+1) \tag{4.52}$$

and their so(3) content consists of the finite-dimensional irreducible so(3) representation with highest weights $M, M-1, \ldots, n$.

5. Representations in the Ω_+ angular momentum basis

In order to obtain the representations ρ' of so(3, 1) in the Ω_+ angular momentum basis we choose as basis for Ω the ordered set

$$\Omega: \{Y(s, n, u, m, t, r), s, n, u, m, t, r \in \mathbb{N}\} \tag{5.1}$$

with $Y(s, n, u, m, t, r) = p_+^s h_+^n p_-^u h_-^m p_3^t h_3^r$, $Y(0, 0, 0, 0, 0, 0) \equiv \mathbb{1}$. The basis for Ω_+ then becomes

$$\Omega_+: \{Y(s, n), s, n \in \mathbb{N}\} \qquad \text{with } Y(s, n) = p_+^s h_+^n. \tag{5.2}$$

It is easily observed that the basis X for Ω goes over into the basis Y of Ω under the Lie algebra automorphism

$$h_3 \to -h_3, \qquad h_+ \to h_-, \qquad h_- \to h_+, \qquad p_3 \to -p_3, \qquad p_+ \to p_-, \qquad p_- \to p_+. \tag{5.3}$$

The representations ρ' of so(3, 1) on Ω and Ω_+ are then obtained from (3.2) and (4.18) by the substitution

$$(\Lambda_1, \Lambda_2) \to (-\Lambda_1, -\Lambda_2) \tag{5.4a}$$

and

$$\rho'(h_3) = -\rho(h_3), \qquad \rho'(h_+) = \rho(h_-), \qquad \rho'(h_-) = \rho(h_+),$$
$$\rho'(p_3) = -\rho(p_3), \qquad \rho'(p_+) = \rho(p_-), \qquad \rho'(p_-) = \rho(p_+). \tag{5.4b}$$

The representations ρ' contain the representations of so(3, 1) which were discussed by Gel'fand *et al* (1963). If one defines

$$l = \Lambda_1 + N - 1, \qquad l_1 = -\mathrm{i}\Lambda_2, \qquad l_0 = \Lambda_1 - 1, \qquad s = \Lambda_1 + N + n \tag{5.5}$$

(s stands here for the m of Gelfand *et al* (1963)) then one obtains the form of the representations as given by Gelfand *et al* (1963) up to an irrelevant redefinition of basis elements. It should be noted, however, that the first of the relations (5.5) is a condition on Λ_1, and thus Λ_1 is no longer arbitrarily complex. Moreover, y_N is a $\rho'(h_-)$ extremal vector for $n = 0$ and $n = -2(\Lambda_1 + N) + 1$. Since $n \geqslant 0$ integer, it follows for the existence of two extremal vectors that $2(\Lambda_1 + N) \leqslant 0$, integer. Since $N \in \mathbb{N}$, it follows that $2\Lambda_1$ must be a negative integer and that the so(3) content of the so(3, 1) representations can contain only a finite number of indecomposable so(3) representations.

Inspection of the representations ρ, equations (4.18) (or the representations ρ', equations (5.4)) shows that they are in fact valid for all $m, N \in \mathbb{Z}$ ($n, N \in \mathbb{Z}$). The subset of y_N^m (y_N^n) for $m, N \in \mathbb{N}$ ($n, N \in \mathbb{N}$) spans an invariant subspace, while the subspace which is spanned by the elements y_N^m (y_N^n) for negative integers is not invariant. While here the extension of the representations to values in $\mathbb{Z}$ is purely formal, it can be shown that this extension actually corresponds to representations of the so(3) subgroup modulo the Casimir element (Gruber and Klimyk 1984).

Having made this formal extension of the representations ρ (and ρ') from $\mathbb{N}$ to $\mathbb{Z}$ it follows that now $2\Lambda_1$ may be an arbitrary integer. This implies that the representations ρ (and ρ') may now contain an infinite number of indecomposable so(3) representations.

We will discuss in the following the representations ρ' in order to establish the connection to the standard so(3, 1) representation theory as presented by Gelfand *et al* (1963) and Naimark (1964).

If in (4.18) the substitution (5.4*a*) is made and the definitions (5.4*b*) are applied, then (4.18) define the representations ρ' on Ω_+. For the extended representation ρ' the basis can formally be chosen as

$$V: \{y_N^n, n \in \mathbb{Z}, N \in \mathbb{N}\} \tag{5.6}$$

where V denotes the linear span of the set of basis elements. The subset

$$\Omega_+: \{y_N^n, n, N \in \mathbb{N}\} \tag{5.7}$$

then spans an so(3, 1) invariant subspace of V, while the subset

$$V-\Omega_+ : \{y_N^n, n\in\mathbb{N}^-, N\in\mathbb{N}\} \tag{5.8}$$

spans a non-invariant subspace of V. The extended representation ρ', however, induces on the quotient space V/Ω_+ a representation (which is in general indecomposable) for which we can choose as basis the set given by (5.8). The standard so(3, 1) representation theory is then obtained on an invariant subspace of the quotient space V/Ω_+ for particular values of the parameters Λ_1, Λ_2.

Figure 5 describes graphically the situation for the case of the representation ρ'. The substitution (5.4a) yields the coefficients α'_N and β'_N of ρ',

$$\alpha'_N=\alpha_N(-\Lambda_1,-\Lambda_2), \qquad \beta'_N=\beta_N(-\Lambda_1,-\Lambda_2). \tag{5.9}$$

For the parameters Λ_1, Λ_2 we choose the values

$$\begin{aligned}&\Lambda_1=l_0+1=\tfrac{1}{2}k, \qquad k-2\in\mathbb{N},\\ &\Lambda_2=\mathrm{i}l_1, \qquad \mathrm{i}\Lambda_2=\tfrac{1}{2}t, \qquad t\in\mathbb{N}, \qquad t=k+2s, \qquad s\in\mathbb{N},\end{aligned} \tag{5.10}$$

and we set

$$l=l_0+N=\Lambda_1+N-1, \qquad s=\Lambda_1+N+n=l+1+n, \qquad n\in\mathbb{Z}. \tag{5.11}$$

The standard so(3, 1) representation theory is represented on the left half of the diagram. In the standard analysis the invariant subspace $C+F$ is invisible since, in the normalisation chosen, it is mapped into zero. The subspaces A and D cannot be seen since the analysis is restricted to the invariant subspace $B+E$ (actually to the quotient space of $B+E+C+F$ modulo $C+F$, which is spanned by the elements of $B+E$). It is only the regions B and E which are considered in the standard analysis.

The elements of E span an invariant subspace of the space which is spanned by the elements of B and E. A basis for this space can be chosen as

$$B+E : \{y_N^{-1}, y_N^{-2}, \ldots, y_N^{-2(\Lambda_1+N)+1}, N\in\mathbb{N}\}. \tag{5.12}$$

This representation is indecomposable for the values of the parameters given by (5.10). It contains an so(3, 1) invariant subspace E with basis

$$E : \{y_{N'+N}^{-1}, y_{N'+N}^{-2}, \ldots, y_{N'+N}^{-2(\Lambda_1+N'+N)+1}, N'=\tfrac{1}{2}(t-k)+1, N\in\mathbb{N}\}, \tag{5.13}$$

$$(B+E)/E : \{y_N^{-1}, y_N^{-2}, \ldots, y_N^{-(\Lambda_1+N)+1}, N=0, 1, \ldots, \tfrac{1}{2}(t-k)\}. \tag{5.14}$$

The representation ρ' induces on the space $(B+E)/E$ the finite-dimensional irreducible so(3, 1) representations of dimension

$$\dim\rho'(t,k)=\tfrac{1}{2}(t+k)(t-k-1)+2k-1. \tag{5.15}$$

The so(3) content of these representations consists of the finite-dimensional so(3) representations

$$l_0=l_1-1=\tfrac{1}{2}k-1, l_0+1, l_0+2, \ldots, \tfrac{1}{2}t-1 \qquad \text{with } k-2\in\mathbb{N} \text{ and } t=k+2s, s\in\mathbb{N}.$$

On the space E the representations ρ' induce infinite-dimensional irreducible representations (the 'tail' of Gelfand *et al* (1963)) whose so(3) content consists of the finite-dimensional irreducible so(3) representations

$$\tfrac{1}{2}t, \qquad \tfrac{1}{2}t+1, \qquad \tfrac{1}{2}t+2, \ldots .$$

The matrix elements for these representations are easily obtained from the extended representation ρ' by limiting the representation ρ' to the respective basis and by setting equal to zero all basis elements which result from the action of ρ' but which do not belong to the basis. These are elements which belong to the invariant subspaces and thus can be set equal to zero.

In addition to the familiar representations of so(3, 1) figure 5 depicts other so(3, 1) representations, indecomposable as well as irreducible ones. For example, the quotient space which is represented by the regions A, B and C carries an indecomposable so(3, 1) representation. This representation has, in turn, two invariant subspaces, namely $B+C$ and C. Restriction to these subspaces yields new representations. Still other representations are obtained on the quotient spaces $A/B+C$, $A+B/C$, etc.

Figures 3 and 4 can be extended in a similar way to represent the extension of the representations ρ to negative integers.

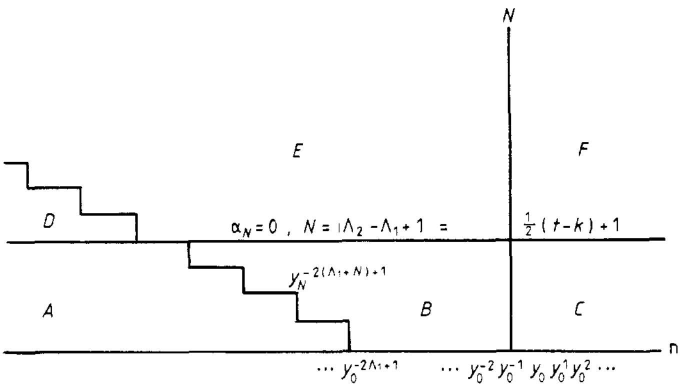

Figure 5. The representations ρ' for the values $\Lambda_1 = \frac{1}{2}k$, $k-2 \in \mathbb{N}$, $i\Lambda_2 = \frac{1}{2}t$, $t \in \mathbb{N}$, $t = k+2s$, $s \in \mathbb{N}$. The full lines characterise the boundaries of invariant subspaces. The invariant subspaces are always to the right of and above the full lines. For a given invariant subspace (regions A, B, etc) the basis elements on the line to the left of and on the bottom line belong to the invariant subspaces.

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